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J.A. SCHOUTEN: A MASTER AT TENSORS
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BY ALBERT NIJENHUIS

1. The name of the Dutch mathematician J.A. Schouten is justifiably associated with tensors. Not only is practically all his work closely related to tensors; it also spans the full range, in many ways:

from the notational and conceptual confusion of the early 20th century to the subtle formalism of his kernel-index method;

from the foundational questions related to Klein's *Erlanger Programm* to the applications to unified field theory and engineering science;

from "safely" imbedded Riemannian manifolds to general affine connections in abstract manifolds;

from simple differential equations to Pfaffian systems and their generalizations.

Schouten's mathematical life is a record of 40 years of hard work, exceptional willpower, a bibliography of almost 200 publications, and numerous distinctions.

2. Jan Arnoldus Schouten was born in Nieuweramstel (now a part of Amsterdam) and spent most of his childhood in Nijmegen. In 1901 he entered the Polytechnical School (re-named *Technische Hoogeschool* in 1905) in Delft and became a student in electrical engineering, a new field in those days. It was here that C.J. Snijders gave him his first exposure to vector analysis. He finished in 1908, and was employed by the city of Rotterdam soon after. The electrification of the city was in progress at that time.

In 1909 Schouten married an old friend. This marriage produced one son and two daughters. It ended in divorce, 34 years later.

In 1912 Schouten gained some financial independence through an inheritance, gave up his job and enrolled in the University of Leiden. His doctoral thesis was started under the direction of J.A. Barrau and was finished under J. Cardinaal. Immediately following he was appointed Professor of Pure and Applied Mathematics and Mechanics at his Alma Mater in Delft.

Schouten's mathematically active life, which begins at this point, can be divided into two periods, the Delft period, which is the longer and more intensive one and ends around 1940, and the Epe-Amsterdam period, which begins around 1943 and ends only a few years before his death.

During the Delft period Schouten writes the great majority of his papers and trains most of his pupils. He explores many areas of differential geometry, which we outline in another section. His work leads to recognition, a distinction at Heidelberg, election to membership of the Royal Netherlands Academy of Sciences, three terms (1929-1931) as president of the Netherlands mathematical society (*Wiskundig Genootschap*) and many trips and lecturing invitations. He assumes extensive administrative responsibilities at Delft; in 1938-1939 he holds the rotating position of *Rector Magnificus*.

When in 1940 the country is overrun by the war, Schouten reaches a crisis. He faces the conflict of loyalty between his German mother whom he had loved dearly and the invader whose mentality he despises; many of the pressures on the universities are felt in Delft, and by him, in particular. There also are mounting domestic problems and failing health. Schouten withdraws into the quiet woods of the village Epe and lives by himself. In 1943 he makes the final break: he resigns his position in Delft and divorces his first wife. Shortly afterwards he re-marries. His new wife helps him regain his health and remains his steady companion for the remaining 28 years of his life.

During the Epe-Amsterdam period the emphasis of Schouten's mathematical life is on Pfaffian systems and, naturally, on a consolidation of what he has already achieved. He resumes traveling, presents papers at a number of international meetings and receives a distinction at Venice. He serves another term (1954) as president of the *Wiskundig Genootschap*, and becomes an honorary member in 1966. In 1954 he is president of the International Congress of Mathematicians, in Amsterdam. His main commitment, however, is to the *Mathematisch Centrum* in Amsterdam, which he serves as Director, member of the Executive Board and as member of the Board of Regents, untill 1968. From 1948 to 1953 he is Extra-ordinary Professor at the University of Amsterdam, charged with the teaching of differential geometry.

Schouten had a forceful personality. Little time was wasted hesitating over decisions, and once made they were adhered to. He refused to indulge in self-pity and had little patience with people who acted otherwise. This directness and strictness intimidated many people; yet he was a sensitive person who knew those around him much better than they expected of this brisk and vigorous man.

Schouten's strictness with himself, his demand for high quality, his careful preparation of lectures and his scrupulously precise writing of papers (including extensive references to work of others); all these were valuable examples to his pupils as they went through their apprenticeship with him. Schouten demanded the same standards of his pupils and rewarded them with increasing responsibilities and involvement in his own research.

Tough as he was, Schouten had a strong sense of fairness, a lively sense of humor and was ready to admit when he was wrong. He had few close friends but even fewer enemies, and was deeply respected by many. Those who got a little closer and enjoyed the generous hospitality at the *Zilvergors* (his home during most of his years in Epe) were given a look at a man who could be tender and kind while still calling a spade a spade.

The combination of firmness and sensitivity was a great asset whenever hard personal decisions had to be made. Schouten proved this over and over, in a great many situations. He succeeded in intimidating a military guard in his fluent German when the T.H. in Delft was closed and so saved valuable papers; he served on a committee which had the delicate task of assigning professorial positions after the war had left everything in chaos; he made the Mathematical Centre a smoothly running organization; he put the long-neglected mathematics library of the University of Amsterdam back on its feet; he headed the organization of the 1954 International Congress; and he was the one who averted disaster when the *Wiskundig Genootschap* was split by a controversy over more democratic government. --- In 1953 he received the well-deserved Royal distinction of *Ridder in de Orde van de Nederlandse Leeuw* for his work in Amsterdam.

Schouten had numerous pupils and co-workers. Some started as pupils and became his colleagues; others came to him after they had ceased to be students. Many have made reputations for themselves. Any list should include the names of D.J. Struik, D. van Dantzig, V. Hlavatý, S. Gołąb, E.R. van Kampen, J. Haantjes, W. van der Kulk, E.J. Post and A. Nijenhuis. His co-authors also include É. Cartan and K. Yano.

3. In the early part of this century the state of vector analysis was one of confusion: there were the systems of Gibbs, Grassmann and Hamilton, there were such quantities as axial and polar vectors, bivectors, quaternions and other "higher number systems". Few mathematicians understood the relations between these competing systems, and most brushed off the whole thing as "engineers' mathematics". In the spirit of his *Erlanger Programm*, which considers geometry as the study of invariants under the action of a group, Felix Klein suggested a study of the groups acting on space, and a classification of all quantities by their behavior under the action of these groups. The execution of this program in 3 dimensions, for tensors up to degree 2, was the subject of Schouten's doctoral dissertation [1914.2].* It was published as a book, and contains an introduction by Felix Klein. Still, Schouten was not satisfied. Its notation was complicated, and he took Hermann Weyl's complaint about "orgies of formalism" quite personally. His search for better notations would take several more years. He experimented with "direct" (i.e. index-free) notations and found them hard to handle: everyone *thought* in terms of components but suppressed them in writing: the reader had the job of mentally reconstructing them. At the advice of Felix Klein, Schouten set himself to design a good notation in which indices were used to best advantage. His final answer was the "kernel-index method".

Meanwhile, Schouten's thesis had little direct effect, but the ideas became known through later papers, which were also based on the classification principle. Other authors, influenced by Lie, Klein or Schouten, also developed their own "direct" methods based on group theory, notably É. Cartan. The interest in this whole area was, naturally, greatly stimulated by Einstein's theory of relativity.

In his large paper [1918.6] Schouten studied the differentiation theory of tensors (he called them "affinors" most of his life) in higher dimensional spaces, and discovered what he called geodesically moving reference systems. Due to the war he had been unaware of Levi-Civita's famous paper (1917) which introduced parallelism. Thus Schouten lost priority for his discovery to Levi-Civita (whose paper was more readable, actually), though Schouten's formulation was independent of any imbedding in Euclidean space, and therefore brought out the parallelism in a Riemannian space as a more intrinsic notion.

As generalizations of the parallelism idea were developed by Weyl (1918) and Eddington (1921), Schouten was led to a classification [1922.1] of all

* *Numbers in square brackets refer to the bibliography at the end of this book.*

linear connections (he used the more dynamic German word "Übertragung") which could be based on the notion of parallelism. This opened the way to a study of spaces with general connections, to which many papers were devoted. These general affine connections are still known under the same name, and are usually not symmetric. In order to maintain the utmost in generality he allowed the co- and contravariant quantities their own connections (hence, contraction and covariant differentiation did not commute); even in his *Ricci Kalkül* [1924.2] he maintains the two independent connections.

The ideas which were developed in the elaboration of connection theory turned out to be extremely fruitful: they became a standard method whose applicability seemed to have no bounds. During the next 16 years Schouten turns out about 100 papers, most of which depend heavily on these ideas. In collaboration with Struik, Van Dantzig, Hlavatý, Golab, Van Kampen and Haantjes the methods are applied to ever new situations: imbeddings and their deformations, conformal geometry, projective geometry, Hermitian and unitary spaces, Kähler manifolds (first discovered by Schouten and Van Dantzig [1931.5] though usually credited to Kähler after his 1933 paper), Lie groups and spinors. Several papers are devoted to unified field theory, too. Also the Lie derivative (discovered by Ślebodziński in 1932) fits exactly into the system.

A new report on the state of the art appears in what has become known as *Einführung I* [1935.5]; it has been the standard textbook for Schouten's later pupils. It deals with the basic concepts and the fundamentals of connection theory from a rather formal point of view. It contains the kernel-index method in its final, polished form, and in its discussion of tensors it makes a clear distinction between the algebraic and analytic properties. The applications to differential geometry appear in *Einführung II* (1938) which was written by D.J. Struik. (Struik had been Schouten's assistant from 1917 to 1924.) Both volumes have been translated into Russian.

Schouten has always remained under the strong influence of the *Erlanger Programm*, even when its original formulation had become insufficient. In [1924.8, 1926.3] he discusses its relations to connection theory; it retains its full validity for the geometry of the tangent space, but not in any small neighborhood in a manifold with a connection. Problems related to the *Erlanger Programm* which Schouten studied extensively include the following:

The classification of tensors. In its full generality this problem has no hope of solution; Schouten succeeded, however, to classify all 3-vectors in 7-dimensional space [1931.1].

The classification of tensors by symmetry types. This cruder approach led to more success; it is recorded in the last chapter of the *Ricci Kalkül* [1924.2]. No comparative study of Schouten's work and that of Young and his tableaux seems to have been made; both are hard to read.

The above two are algebraic problems: they concern the tangent space to a manifold at one point. In contrast to this is the search for differential (con)comitants of tensor fields; that is, for tensor fields whose components are invariant functions of the components of given tensor fields *and* their derivatives. Lie brackets of vector fields, Lie derivatives, and the curl of a differential form (covariant multi-vector field) are the common examples; the Riemann-Christoffel tensor as functional of the metric tensor field is still the most bafflingly ingenious construction of all. Schouten searched for more differential concomitants besides these and some other known ones; he initially found only a few [1940.5], and later a few more that appear as exercises in [1954.1]; see also [1953.2]. His last few papers [1956.2] extend work on another differential concomitant found by his assistant Nijenhuis (1951, 1955).

A variation on the above is the search for complete sets of differential invariants of one or more tensor fields. The problem was solved, in the presence of a connection, by Veblen (1922) and his students in their studies of the geometry of paths and normal coordinates. The general problem without a connection seems out of reach again. The special case of a single covariant vector field led him and Van der Kulk to their study of Pfaff's problem and generalizations. Their findings have been recorded in many papers and have been collected in [1949.1].

Of course, the whole differentiation theory of tensor fields and their geometry is a search for differential concomitants of tensor fields and/or connections. Schouten has worked on this problem most of his life, really: his kernel-index method was an ideal tool to ascertain the transformation properties of any expression he came across, and this is no co-incidence.

Also, Schouten's interest in unified field theory was linked to the *Erlanger Programm*: the basic fields (electrical, gravitational, etc.) are largely independent of each other, and have their own invariance properties. Ideas along this line have been continued by E.J. Post.

In the last few years before his retirement Schouten wrote two books, both containing an exposition of his methods. His *"Tensor analysis for physicists"* [1951.3] is more sophisticated than the great majority of books on the subject and aimed at a selective audience. His *"Ricci Calculus"*

[1954.1], officially a new edition of [1924.2] is, in fact, an entirely new book and a veritable storehouse of information on the state of local differential geometry at that time. (The involvement in its writing has been an unforgettable apprenticeship for the author of this article.)

Together with *"Pfaff's Problem"* these books are a record of the state of local differential geometry as Schouten helped shape it and saw it at the conclusion of his fruitful career.

Editors' note: The remaining part of this biography contained a bibliography of Schouten's work. It is not reproduced here, because a revised and extended version of that bibliography is printed at the end of this book. Where necessary, the numbering of the references in the above article has been adapted.